# Orbital angular momentum of Laguerre–Gaussian beams beyond the paraxial approximation

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We derive a full field solution for Laguerre–Gaussian beams consistent with the Helmholtz equation using the angular spectrum method. Field components are presented as an order expansion in the ratio of the wavelength to the beam waist, $f = \lambda/(2\pi w_0)$, which is typically small. The result is then generalized to a beam of arbitrary polarization. This result is then used to reproduce the signature angular momentum properties of Laguerre–Gaussian beams in the paraxial limit. The subsequent higher-order term is similarly obtained, which does not display a clear separation of orbital and spin angular momentum components. 

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## 1. INTRODUCTION

Nearly two decades ago, Allen *et al.* demonstrated that cylindrical solutions to the paraxial wave equation, Laguerre-Gaussian beams, possess a well-defined angular momentum, which can be specified in terms of an orbital component due to the shape of the electromagnetic field and a spin component due to the polarization of the beam [1]. While it had been known that light was able to impart angular momentum in objects due to its polarization [2], this was the first theoretical result demonstrating both the orbital and spin contributions to this effect. These results have also had significant experimental implications, most notably in optical trapping techniques. Less than a decade ago, O'Neil *et al.* experimentally observed both the transfer of orbital and spin angular momentum to a particle trapped in an otherwise traditional optical trap [3]. This is an exciting experimental frontier since it can open new ways of manipulating trapped particles, enabling a wide array of experiments in physics and biology.

Since these first theoretical investigations, much effort has been expended to understand the general definition and role of orbital and spin angular momentum beyond the paraxial limit. Initial efforts focused on decomposing the Poynting vector into orbital and spin components [4,5]. Subsequently, it was noted that the conventional choice for general field polarizations was incompatible with the requirement that fields be transverse in momentum space, and this led to the discovery of a spin–orbit interaction [6,7]. This interaction predicts that nonparaxial corrections to the orbital angular momentum will depend upon $\sigma$, the parameter that characterizes the spin angular momentum of plane waves.

The angular spectrum method, an expansion of the Green's function in free space, is well known. An excellent review of this method was written by Bouwkamp [8] with early applications appearing in Clemmow [9]. In this paper, we use the angular spectrum method to find the full field solutions for Laguerre–Gaussian beams with arbitrary polarization. Our approach is similar to previous authors, who used the angular spectrum method to find the exact field solutions of $\mathrm{TEM}_{00}$ laser modes in powers of the small parameter $f = 1/(kw_0)$, where $w_0$ is the beam waist at the focal spot [11–14]. This small parameter, $f$, was first introduced by Lax *et al.* in their analysis of how the paraxial equation, and corrections to it, arise naturally and consistently from an order expansion of the source-free Maxwell equations [15].

A previous attempt to use the angular spectrum method to derive the electric field components of Laguerre–Gaussian beams was performed by Chen *et al.* [14]. This work demonstrated that the spherical Bessel functions generated using the angular spectrum method on Laguerre–Gaussian beams correctly reproduced the waist function seen in paraxial representations of the fields. These results, which extend the work of Agrawal and Pattanayak [11], demonstrate that the divergence seen in the longitudinal direction of the field components is the result of interchanging summations. This result led to increased activity in deriving the far-field expressions for Laguerre–Gaussian beams using singular expansion techniques [16] or the method of stationary phase [17].

In this paper we derive the full field components for Laguerre–Gaussian beams of arbitrary polarization using the angular spectrum method. We stress that the field components presented here are exact solutions to the full Helmholtz equation, expressed in powers of $f$, with the evanescent components removed. With the full field components we then demonstrate that by keeping only the paraxial terms, $O(f^2) \to 0$, the quantization of the angular momentum of Laguerre–Gaussian beams separates into orbital and spin components, as expected, but when keeping $O(f^2)$ the separation into orbital and spin components disappears [7]. Furthermore, this leading order, nonparaxial angular momentum derived here does not display the same longitudinal direction divergences seen in the field intensities noticed by Chen *et al.* [14].

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In the following section, the angular spectrum method is used to develop the field components given Laguerre–Gaussian beam initial conditions. The next section presents an error analysis of the integral evaluations and a discussion of the source of the longitudinal divergence. The full field components are then derived in the third section. The full expression for the field angular momentum is developed in the fourth section and the discussion concludes with a summary of the possible applications of these results.

## 2. ANGULAR SPECTRUM METHOD

For a TE Laguerre–Gaussian beam with linear polarization along the $x$ direction, we have $E_y = 0$ for all space and

$$E_x(x,y,0) = E_0\left(\frac{\sqrt{2}\rho}{w_0}\right)^l L_p^{(l)}\left(\frac{2\rho^2}{w_0^2}\right)\exp\left[-\frac{\rho^2}{w_0^2}\right]\exp[il\phi] \quad (1)$$

inside the aperture, where $L_p^{(l)}$ is the associated Laguerre polynomial, $\rho$ is $\sqrt{x^2+y^2}$, $\phi = \arctan(y/x)$, $E_0$ is the initial field amplitude, and $w_0$ is the beam waist radius. An implicit harmonic time dependence of $\exp[-i\omega t]$ has been suppressed. The amplitude function of $E_x$ in general is given by [8,9]

$$A(b,\theta) = E_0\left(\frac{k}{2\pi}\right)^2\int_0^\infty\int_0^{2\pi} E_x(x,y,0)\exp[-ik\mathbf{b}\cdot\boldsymbol{\rho}]\rho\mathrm{d}\rho\mathrm{d}\phi, \quad (2)$$

where $\mathbf{b}\cdot\boldsymbol{\rho}$ represents the usual angular spectrum variables $b_x x + b_y y = b\rho\cos(\phi-\theta)$, $b^2 = b_x^2 + b_y^2$ and $\theta = \arctan(b_y/b_x)$. Inserting the initial conditions produces

$$A(b,\theta) = E_0\left(\frac{k}{2\pi}\right)^2\int_0^\infty\int_0^{2\pi}\left(\frac{\sqrt{2}\rho}{w_0}\right)^l L_p^{(l)}\left(\frac{2\rho^2}{w_0^2}\right) \times\exp\left[-\frac{\rho^2}{w_0^2}\right]\exp[il\phi] \times\exp[-ikb\rho\cos(\phi-\theta)]\rho\mathrm{d}\rho\mathrm{d}\phi, \quad (3)$$

which is the amplitude function for a specified $(p,l)$ Laguerre–Gaussian mode.

The angular portion of this integral can be evaluated by using the well-known identity

$$\int_0^{2\pi}\exp[il\phi + ikb\rho\cos(\phi-\theta+\pi)]\mathrm{d}\phi = 2\pi\exp\left[-i\frac{\pi}{2}l\right]J_l(kb\rho)\exp[il\theta], \quad (4)$$

where $J_l$ is the Bessel function of the first kind of order $l$. The Laguerre polynomial can also be expanded as

$$L_p^{(l)}\left(\frac{2\rho^2}{w_0^2}\right) = \sum_{\alpha=0}^{p}(-1)^\alpha\binom{p+l}{p-\alpha}\frac{1}{\alpha!}\left(\frac{\sqrt{2}\rho}{w_0}\right)^{2\alpha}, \quad (5)$$

which allows us to rewrite our expression for the amplitude function as

$$A(b,\theta) = E_0\frac{k^2}{2\pi}\exp[il\theta]\exp\left[-il\frac{\pi}{2}\right]\sum_{\alpha=0}^{p}(-1)^\alpha\binom{p+l}{p-\alpha}\frac{1}{\alpha!} \times\int_0^\infty\left(\frac{\sqrt{2}\rho}{w_0}\right)^{2\alpha+l}\exp\left[-\frac{\rho^2}{w_0^2}\right]J_l(kb\rho)\rho\mathrm{d}\rho. \quad (6)$$

This integral has a known closed form solution [18], which is

$$A(b,\theta) = \frac{E_0}{2\pi f^2}\exp[il\theta]\exp\left[-il\frac{\pi}{2}\right] \times\sum_{\alpha=0}^{p}(-1)^\alpha\binom{p+l}{p-\alpha}(\sqrt{2})^{2\alpha+l-2} \times\left(\frac{b}{2f}\right)^l\exp\left[-\frac{b^2}{4f^2}\right]L_\alpha^{(l)}\left(\frac{b^2}{4f^2}\right), \quad (7)$$

where $f = 1/(kw_0) = \lambda/(2\pi w_0)$ is the natural expansion factor appearing in Lax *et al.* [15].

There are a few things that are immediately interesting about this formula. First, in agreement with Zhou [17], the angular momentum term has been preserved in amplitude space. Furthermore, this formula is exact—no approximation has been introduced in the derivation.

With the generalized amplitude of the Laguerre–Gaussian beam in hand, we can construct the complete field of the vector component $E_x$ as

$$E_x(x,y,z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}A(b,\theta)\exp[ik(\mathbf{b}\cdot\boldsymbol{\rho}+mz)]\mathrm{d}^2\mathbf{b}, \quad (8)$$

where $m^2 = 1-b^2$. The positive root of $m$ is taken since this corresponds to waves propagating in the $+\hat{z}$ direction. Although this expansion holds for all real values of $b$, it is a useful, and common, approximation to ignore the evanescent waves [12,13], which corresponds to $b^2 > 1$ so that $A(b,\theta) = 0$ for these values. Thus, the complete spatial dependence of $E_x$ becomes

$$E_x(x,y,z) = \frac{E_0}{2\pi f^2}\exp\left[-il\frac{\pi}{2}\right]\sum_{\alpha=0}^{p}(-1)^\alpha\binom{p+l}{p-\alpha}(\sqrt{2})^{2\alpha+l-2} \times\int_0^1\int_0^{2\pi}\left(\frac{b}{2f}\right)^l\exp\left[-\frac{b^2}{4f^2}\right]L_\alpha^{(l)}\left(\frac{b^2}{4f^2}\right) \times\exp[il\theta]\exp[ik\mathbf{b}\cdot\boldsymbol{\rho}]\exp[ikmz]b\mathrm{d}b\mathrm{d}\theta. \quad (9)$$

The angular integral can be evaluated in the same way as above. To evaluate the radial integral, decompose the associated Laguerre polynomial again

$$L_\alpha^{(l)}\left(\frac{b^2}{4f^2}\right) = \sum_{\beta=0}^{\alpha}(-1)^\beta\binom{\alpha+l}{\alpha-\beta}\frac{1}{\beta!}\left(\frac{b}{2f}\right)^{2\beta}, \quad (10)$$

to retain only the explicit polynomial terms. Using the multiplication relations for Bessel and modified Bessel functions [11,12,19], $\exp[ikmz]$ can be expressed as an exact summation in the form

$$\exp\left[ikz\sqrt{1-b^2}\right] = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{b^2}{2}\right)^n(kz)^{n+1}h_{n-1}^{(1)}(kz), \quad (11)$$

since $b^2 < 1$ and where $h_n^{(1)}$ is the $n$th-order spherical Bessel function of the third kind. With these expansions, and after evaluating the angular integral, the field $E_x$ can be expressed as

$$E_x(x,y,z) = E_0 \exp[il\phi] \sum_{n=0}^{\infty} (kz)^{n+1} h_{n-1}^{(1)}(kz) \times \sum_{\alpha=0}^{p} \sum_{\beta=0}^{\alpha} (-1)^{\alpha+\beta} \binom{p+l}{p-\alpha} \binom{\alpha+l}{\alpha-\beta} \times \frac{1}{n!\beta!} I_{n,l,\alpha,\beta}(\rho), \tag{12}$$

where

$$I_{n,l,\alpha,\beta}(\rho) = (\sqrt{2})^{2\alpha+2n+l-2} \frac{1}{f^{2\beta+l+2}} \times \int_0^1 \left(\frac{b}{2}\right)^{l+2n+2\beta} \exp\left[-\frac{b^2}{4f^2}\right] J_l(kb\rho) b \mathrm{d}b. \tag{13}$$

This last integral may now be expressed as the difference between two infinite-range integrals

$$I_{n,l,\alpha,\beta}(\rho) = (\sqrt{2})^{2\alpha+2n+l-2} \frac{1}{f^{2\beta+l+2}} \times \left( \int_0^\infty \left(\frac{b}{2}\right)^{l+2n+2\beta} \exp\left[-\frac{b^2}{4f^2}\right] J_l(kb\rho) b \mathrm{d}b - \int_1^\infty \left(\frac{b}{2}\right)^{l+2n+2\beta} \exp\left[-\frac{b^2}{4f^2}\right] J_l(kb\rho) b \mathrm{d}b \right), \tag{14}$$

each of which converges absolutely for fixed $n$. It is clear from the form of these integrals that the large $b$ behavior is controlled by the Gaussian exponential term, especially because $1/f^2$ is large. For certain ranges of $f$ with fixed values of $l$ and $p$, the second integral may be neglected as has been previously demonstrated [11,12,20]. A thorough discussion of when this is possible is presented in a section below, but for now we will proceed assuming that we are within the acceptable parameter space.

After discarding this second integral, $I(\rho)$ can be evaluated as [18]

$$I_{n,l,\alpha,\beta}(\rho) = f^{2n} 2^{n+\alpha} (n+\beta)! \left(\frac{\sqrt{2}\rho}{w_0}\right)^l L_{n+\beta}^{(l)}\left(\frac{\rho^2}{w_0^2}\right) \exp\left[-\frac{\rho^2}{w_0^2}\right]. \tag{15}$$

With this result in hand, the complete spatial dependence of $E_x$ has the form

$$E_x(x,y,z) = E_0 \sum_{n=0}^{\infty} f^{2n} (kz)^{n+1} h_{n-1}^{(1)}(kz) \times \sum_{\alpha=0}^{p} \sum_{\beta=0}^{\alpha} (-1)^{\alpha+\beta} \binom{p+l}{p-\alpha} \binom{\alpha+l}{\alpha-\beta} \binom{n+\beta}{n} \times 2^{n+\alpha} \left(\frac{\sqrt{2}\rho}{w_0}\right)^l L_{n+\beta}^{(l)}\left(\frac{\rho^2}{w_0^2}\right) \times \exp\left[-\frac{\rho^2}{w_0^2}\right] \exp[il\phi]. \tag{16}$$

This last expression represents a significantly cleaner result for Laguerre–Gaussian beams using the angular spectrum method than has previously been presented [17,20]. Note that the field component appears as an expansion in powers of the natural order parameter $f$. Most importantly, it makes obvious the conservation of the angular momentum of the Laguerre–Gaussian beam through the transformations that comprise the angular spectrum method. It should again be emphasized that no approximation has been made in the derivation of this equation. So long as the second integral in Eq. (14) can be neglected, which depends on $p$, $l$, and $f$, this expression represents an exact solution to the full Helmholtz equation. Additionally, the restricted dependence upon the even powers of $f$ is in agreement with Lax *et al.* [15].

## 3. ERROR INTEGRAL ANALYSIS

The results presented here are only valid when the second integral in Eq. (14) vanishes sufficiently fast. Rewriting this second integral

$$I_{\mathrm{error}}(r) = (\sqrt{2})^{2\alpha+2n+l-2} \frac{1}{f^{2\beta+l+2}} \int_1^\infty \left(\frac{b}{2}\right)^{l+2n+2\beta} \times \exp\left[-\frac{b^2}{4f^2}\right] J_l(kbr) b \mathrm{d}b \le (\sqrt{2})^{2p+2n+l-2} \frac{1}{f^{2p+l+2}} \times \int_1^\infty \left(\frac{b}{2}\right)^{l+2n+2p} \exp\left[-\frac{b^2}{4f^2}\right] b \mathrm{d}b, \tag{17}$$

where $\alpha$ and $\beta$ have been set equal to $p$, which is the value they have for the largest term in the sum, and the Bessel function has been dropped since it is at most of order unity. By making the substitution $u = b^2$, Eq. (17) can be expressed in terms of the well-known exponential integrals, $E_n(x)$,

$$I_{\mathrm{error}} = \frac{1}{2^{p+n+l/2} f^{p+l/2+1}} E_{-n-p-l/2}\left(\frac{1}{4f^2}\right), \tag{18}$$

which are readily evaluated, even for half-integer order.

Finally, one might suspect that for large $n$ this integral will diverge as $u^n$ begins to shift the maximum of the integrand completely into the range of $I_{\mathrm{error}}$. However, noting the factor of $1/n!$ in Eq. (12), as $n$ becomes large this term behaves as $(u/n)^n$, which tends to zero for large $n$, small $u$. When $u \ge n$, $u \gg 4f^2$ and the exponential term $\exp[-u/4f^2]$ dominates the behavior. Thus, such a shift of the maximum of the integrand due to increasing $n$ does not occur and it can be shown that in fact the error decreases as $n$ increases due to the reasons mentioned above.

For the trivial case with $p = l = 0$, $f \le 0.2$ allows for accuracy greater than $10^{-2}$. This result is in agreement with previously published results [12,20], noting the difference in the definition of the beam waist from these previous results, $w_0 = \sqrt{2} w_{\mathrm{prev}}$. For reasonable values of $p + l/2 \le 15$, we require $f \le 0.055$ for results accurate to within $10^{-2}$.

It should be recognized, though, that this favorable series convergence fails in the limit of large spherical radius, as discussed by Chen *et al.* [14]. Physically, the asymptotic form of the cylindrical field expansion must match a spherically expanding radial plane wave. The onset of this transition region can be quantified by minimizing the phase difference between the cylindrical and spherical representations. Qualitatively, this behavior may be captured by a simple asymptotic expansion of the $E_x$ field component in Eq. (9). That is, observe that the large $kz$ form of Eq. (11) is

$$\exp\left[ikz\sqrt{1-b^2}\right] \approx \exp\left[ikz - ikz\frac{b^2}{2}\right], \tag{19}$$

using the well-known asymptotic form for the spherical Bessel functions and resuming the resultant series. The relevant integral in Eq. (9) becomes

$$\int_0^1 \left(\frac{b}{2}\right)^{l+2\beta} \exp\left[-b^2\left(\frac{1}{4f^2}+\frac{ikz}{2}\right)\right] J_l(k\rho b) b\mathrm{d}b \tag{20}$$

in the large $kz$ limit, which is readily approximated by the method of stationary phase to be

$$\left(\frac{1}{2ikz}\right)\left(\frac{1}{2}\right)^{l+2\beta} \exp[ikz] \exp\left[-\left(\frac{1}{4f^2}+i\frac{kz}{2}\right)\right] J_l(k\rho) \sim \exp\left[i\frac{kz}{2}\right]\frac{J_l(k\rho)}{2ikz} \tag{21}$$

to first order in $(1/kz)$. This simple form clearly displays the expected far-field behavior of the $E_x$ component, indicating that retention of the infinite series is necessary to obtain the appropriate asymptotic spherical form [14].

## 4. FIELD COMPONENTS

With the explicit form of $E_x$ for linearly polarized TE modes known, the rest of the field components for such modes can now be calculated following [11,12] using

$$E_z(x,y,z) = \frac{i}{k}\frac{\partial}{\partial x}\left(\frac{\mathcal{I}}{m}\right), \tag{22}$$

$$\mathbf{B}(x,y,z) = \frac{1}{\omega k}\left[\frac{\partial^2}{\partial x \partial y}\left(\frac{\mathcal{I}}{m}\right)\hat{x} + \left(k^2 m\mathcal{I} - \frac{\partial^2}{\partial x^2}\left(\frac{\mathcal{I}}{m}\right)\right)\hat{y} + ik\frac{\partial}{\partial y}(\mathcal{I})\hat{z}\right], \tag{23}$$

where

$$\mathcal{I} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A(b,\theta)\exp[ik(\mathbf{b}\cdot\boldsymbol{\rho} + mz)]\mathrm{d}^2\mathbf{b}, \tag{24}$$

$$\frac{\mathcal{I}}{m} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A(b,\theta)\frac{\exp[ik(\mathbf{b}\cdot\boldsymbol{\rho} + mz)]}{m}\mathrm{d}^2\mathbf{b}, \tag{25}$$

$$m\mathcal{I} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A(b,\theta) m \exp[ik(\mathbf{b}\cdot\boldsymbol{\rho} + mz)]\mathrm{d}^2\mathbf{b}. \tag{26}$$

These integrals can be evaluated using

$$\frac{\exp\left[ikz\sqrt{1-b^2}\right]}{\sqrt{1-b^2}} = i\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{b^2}{2}\right)^n (kz)^{n+1} h_n^{(1)}(kz), \tag{27}$$

$$\sqrt{1-b^2}\exp\left[ikz\sqrt{1-b^2}\right] = i\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{b^2}{2}\right)^n (kz)^n [kz h_n^{(1)}(kz) - 2n h_{n-1}^{(1)}(kz)], \tag{28}$$

which are expansions similar to Eq. (11) but tailored to match the required integrals above. Even though these expansions differ slightly in their dependence upon $z$, they still have exactly the same factors of $1/n!$ and $(b^2/2)^n$ that were used to evaluate the previous integral above. Thus, they will lead to the same expression for $I_{n,l,\alpha,\beta}(\rho)$ and will converge for the same values of $f$, $l$, and $p$.

The treatment of the field components can be extended to TE modes of arbitrary polarization. The angular momentum of an electromagnetic wave about the axis of propagation due to its polarization is usually defined as $\sigma_z = i(\alpha_z\beta_z^* - \beta_z\alpha_z^*)$ [21]. It should be noted, though, that this definition of field polarization is only correct when the longitudinal component of the total electric field vanishes; otherwise, unambiguous polarization states cannot be defined and a transformed set of field components must be used, as emphasized recently by Li and co-workers [22]. The electric field is given by

$$\mathbf{E}(x,y,z) = (\alpha_z\hat{x} + \beta_z\hat{y})E(x,y,z) + E_z(x,y,z)\hat{z}, \tag{29}$$

where $E(x,y,z)$ is defined similarly to Eq. (1). The polarization factors, $\alpha_z$ and $\beta_z$, are normalized phase factors, such that $|\alpha_z|^2 + |\beta_z|^2 = 1$, which allows for $\sigma_z = \pm 1$ for left- and right-circular polarizations, respectively, as well as $\sigma_z = 0$ for linear polarization, and $0 < |\sigma_z| < 1$ for elliptical polarizations. Finally, $E_z$ is chosen to ensure that the divergence of the total field is zero.

The angular spectrum method is based on an integral transform, which acts as a linear operator. Thus, it is straightforward to generalize it to beams of arbitrary polarization. The field components required by $E_x$, given by Eqs. (22) and (23), are multiplied through by $\alpha_z$, and added to the similar set of equations needed to ensure the field equations are satisfied when there is also a $\hat{y}$ component in the aperture. Thus, for a field of the form of Eq. (29), the exact, full field components generated using the angular spectrum method are given by

$$E_z(x,y,z) = \alpha_z\frac{i}{k}\frac{\partial}{\partial x}\left(\frac{\mathcal{I}}{m}\right) + \beta_z\frac{i}{k}\frac{\partial}{\partial y}\left(\frac{\mathcal{I}}{m}\right), \tag{30}$$

$$\mathbf{B}(x,y,z) = \alpha_z\frac{1}{\omega k}\left[\frac{\partial^2}{\partial x\partial y}\left(\frac{\mathcal{I}}{m}\right)\hat{x} + \left(k^2 m\mathcal{I} - \frac{\partial^2}{\partial x^2}\left(\frac{\mathcal{I}}{m}\right)\right)\hat{y} + ik\frac{\partial}{\partial y}(\mathcal{I})\hat{z}\right] + \beta_z\frac{1}{\omega k}\left[\left(\frac{\partial^2}{\partial y^2}\left(\frac{\mathcal{I}}{m}\right) - k^2 m\mathcal{I}\right)\hat{x} - \frac{\partial^2}{\partial x\partial y}\left(\frac{\mathcal{I}}{m}\right)\hat{y} - ik\frac{\partial}{\partial x}(\mathcal{I})\hat{z}\right]. \tag{31}$$

Finally, we define

$$v_{0,n}(\rho) = E_0 \left(\frac{\sqrt{2}\rho}{w_0}\right)^l \exp\left[-\frac{\rho^2}{w_0^2}\right] \times \sum_{\alpha=0}^{p}\sum_{\beta=0}^{\alpha}(-1)^{\alpha+\beta}\binom{p+l}{p-\alpha}\binom{\alpha+l}{\alpha-\beta}\binom{n+\beta}{n} \times 2^{n+\alpha}L_{n+\beta}^{(l)}\left(\frac{\rho^2}{w_0^2}\right), \tag{32}$$

$$v_{1,n}(\rho) = E_0 \left(\frac{\sqrt{2}\rho}{w_0}\right)^l \exp\left[-\frac{\rho^2}{w_0^2}\right] \times \sum_{\alpha=0}^{p}\sum_{\beta=0}^{\alpha}(-1)^{\alpha+\beta}\binom{p+l}{p-\alpha}\binom{\alpha+l}{\alpha-\beta}\binom{n+\beta}{n} \times 2^{n+\alpha}\frac{l+1}{n+l+\beta+1}L_{n+\beta}^{(l+1)}\left(\frac{\rho^2}{w_0^2}\right), \tag{33}$$

$$v_{2,n}(r) = E_0 \left(\frac{\sqrt{2}\rho}{w_0}\right)^l \exp\left[-\frac{\rho^2}{w_0^2}\right] \times \sum_{\alpha=0}^{p}\sum_{\beta=0}^{\alpha}(-1)^{\alpha+\beta}\binom{p+l}{p-\alpha}\binom{\alpha+l}{\alpha-\beta}\binom{n+\beta}{n} \times 2^{n+\alpha}\frac{(l+1)(l+2)}{(n+l+\beta+1)(n+l+\beta+2)} \times L_{n+\beta}^{(l+2)}\left(\frac{\rho^2}{w_0^2}\right), \tag{34}$$

which are related to radial derivatives of Eq. (16), for simplicity. Given these, the field components of a TE Laguerre-Gaussian beam of arbitrary polarization, where $E(x,y,z)$ in Eq. (29) is given by Eq. (1), are

$$E_x(x,y,z) = \alpha_z \exp[il\phi]\sum_{n=0}^{\infty} f^{2n}(kz)^{n+1}h_{n-1}^{(1)}(kz)v_{0,n}(\rho), \tag{35}$$

$$E_y(x,y,z) = \beta_z \exp[il\phi]\sum_{n=0}^{\infty} f^{2n}(kz)^{n+1}h_{n-1}^{(1)}(kz)v_{0,n}(\rho), \tag{36}$$

$$E_z(x,y,z) = \exp[il\phi]\sum_{n=0}^{\infty} f^{2n+1}(kz)^{n+1}h_{n}^{(1)}(kz) \times \left[(\alpha_z + i\beta_z)lw_0\left(\frac{iy-x}{\rho^2}\right)v_{0,n}(\rho) + \left(\alpha_z\frac{2x}{w_0} + \beta_z\frac{2y}{w_0}\right)v_{1,n}(\rho)\right], \tag{37}$$

$$B_x(x,y,z) = \frac{ik}{\omega}\beta_z \exp[il\phi]\sum_{n=0}^{\infty} f^{2n}(kz)^n\left[2nh_{n-1}^{(1)}(kz) - kzh_n^{(1)}(kz)\right]v_{0,n}(\rho) + \frac{ik}{\omega}\exp[il\phi]\sum_{n=0}^{\infty} f^{2n+2}(kz)^{n+1}h_n^{(1)}(kz) \times \left[\left(\alpha_z\frac{4xy}{w_0^2} + \beta_z\frac{4y^2}{w_0^2}\right)v_{2,n}(\rho) + (\alpha_z + i\beta_z)\frac{l(l-1)w_0^2}{\rho^4}(ix^2 + 2xy - iy^2)v_{0,n}(\rho) - \alpha_z\frac{2l}{\rho^2}(ix^2 + 2xy - iy^2)v_{1,n}(\rho) - \beta_z\left(\frac{4l}{\rho^2}(y^2 + ixy) + 2\right)v_{1,n}(\rho)\right], \tag{38}$$

$$B_y(x,y,z) = \frac{ik}{\omega}\alpha_z \exp[il\phi]\sum_{n=0}^{\infty} f^{2n}(kz)^n\left[kzh_n^{(1)}(kz) - 2nh_{n-1}^{(1)}(kz)\right]v_{0,n}(\rho) - \frac{ik}{\omega}\exp[il\phi]\sum_{n=0}^{\infty} f^{2n+2}(kz)^{n+1}h_n^{(1)}(kz) \times \left[\left(\alpha_z\frac{4x^2}{w_0^2} + \beta_z\frac{4xy}{w_0^2}\right)v_{2,n}(\rho) + (\beta_z - i\alpha_z)\frac{l(l-1)w_0^2}{\rho^4}(ix^2 + 2xy - iy^2)v_{0,n}(\rho) - \beta_z\frac{2l}{\rho^2}(ix^2 + 2xy - iy^2)v_{1,n}(\rho) + \alpha_z\left(\frac{4l}{\rho^2}(ixy - x^2) - 2\right)v_{1,n}(\rho)\right], \tag{39}$$

$$B_z(x,y,z) = \frac{ik}{\omega}\exp[il\phi]\sum_{n=0}^{\infty} f^{2n+1}(kz)^{n+1}h_{n-1}^{(1)}(kz) \times \left[(i\alpha_z - \beta_z)lw_0\left(\frac{x-iy}{\rho^2}\right)v_{0,n}(\rho) + \left(\beta_z\frac{2x}{w_0} - \alpha_z\frac{2y}{w_0}\right)v_{1,n}(\rho)\right]. \tag{40}$$

Note that the field components perpendicular to the direction of propagation are an expansion in even powers of $f$, while the longitudinal components are an expansion in odd powers of $f$, in agreement with Lax *et al.* [15]. Comparing these results with those derived earlier [4,21], we note that factors of the form $x \pm iy$ are related to $\exp[\pm i\phi]$, generated from taking transverse coordinate derivatives of $E(x,y,z)$ in Eq. (29), so there is actually no discrepancy with these previous expressions. Finally, as the dependence upon spherical Bessel functions is similar to that in previous works [14], these formulas could be rearranged to resemble the waist function in paraxial representations of the Laguerre–Gaussian fields [21].

A nearly identical derivation also suffices to define the TM modes of arbitrary polarization for Laguerre–Gaussian beams. A full treatment of these modes is omitted here but is straightforward.

## 5. ANGULAR MOMENTUM

With the components of exact Laguerre–Gaussian beams worked out above, it is important to reproduce their signature angular momentum properties. In the paraxial limit, Laguerre–Gaussian beams have quantized orbital angular momentum proportional to $l$ and spin angular momentum proportional to $\sigma_z$ [1]. Beyond the paraxial limit, the spin–orbit interaction predicts that later terms in the total angular momentum will not be as readily separated and specifically that the orbital angular momentum picks up a term proportional to $\sigma_z$.

With the field components presented above, we can recover the paraxial result by letting $O(f^2) \to 0$. This corresponds to keeping only the first, $n = 0$, term in each sum. One identity that is of some use can be derived from noting that when performing the angular spectrum method and expanding $\exp[ikmz]$ using the Bessel function multiplication relations, the $n = 0$ term corresponds with simply taking two Fourier transforms of the initial function and adding plane wave $z$ dependence. Noting this, it is demonstrable that [23]

$$L_p^{(l)}\left(\frac{cr^2}{w_0^2}\right) = \sum_{\alpha=0}^{p}\sum_{\beta=0}^{\alpha}(-1)^{\alpha+\beta}\binom{p+l}{p-\alpha}\binom{\alpha+l}{\alpha-\beta}c^\alpha L_\beta^{(l)}\left(\frac{r^2}{w_0^2}\right), \tag{41}$$

which allows us to greatly simplify the $n = 0$ terms. To compare our results to Allen *et al.* we define

$$u_j(\mathbf{r}) = E_0 \exp[ikz]\exp[il\phi]\left(\frac{\sqrt{2}\rho}{w_0}\right)^l \exp\left[-\frac{\rho^2}{w_0^2}\right] L_{p-j}^{(l+j)}\left(\frac{2\rho^2}{w_0^2}\right), \tag{42}$$

and note that $|u|^2$ is equivalent to the similarly defined function [24] where the beam waist has taken on a fixed value.

The paraxial terms of the full field components can thus be written as

$$E_x(x,y,z) = \alpha_z u_0(\mathbf{r}), \tag{43}$$

$$E_y(x,y,z) = \beta_z u_0(\mathbf{r}), \tag{44}$$

$$E_z(x,y,z) = if\left[(\alpha_z + i\beta_z)\frac{lw_0(x-iy)}{\rho^2}u_0(\mathbf{r}) - \frac{2}{w_0}(\alpha_z x + \beta_z y)u_0(\mathbf{r}) - \frac{4}{w_0}(\alpha_z x + \beta_z y)u_1(\mathbf{r})\right], \tag{45}$$

$$B_x(x,y,z) = -\beta_z \frac{k}{\omega} u_0(\mathbf{r}), \tag{46}$$

$$B_y(x,y,z) = \alpha_z \frac{k}{\omega} u_0(\mathbf{r}), \tag{47}$$

$$B_z(x,y,z) = if\frac{k}{\omega}\left[(\alpha_z + i\beta_z)\frac{lw_0(y+ix)}{\rho^2}u_0(\mathbf{r}) - \frac{2}{w_0}(\alpha_z y - \beta_z x)u_0(\mathbf{r}) - \frac{4}{w_0}(\alpha_z y - \beta_z x)u_1(\mathbf{r})\right], \tag{48}$$

where the spherical Bessel functions have been rewritten as exponentials and then collapsed into the factors of $u$.

The energy flux density is given by the real part of the Poynting vector, which is given by $\mathbf{S} = 1/2[\mathbf{E}\times\mathbf{H}^* + \mathbf{E}^*\times\mathbf{H}]$. For paraxial Laguerre–Gaussian beams, this is

$$S_x = \sigma_z \frac{fk}{\mu\omega}\left[\left(\frac{lw_0 y}{\rho^2} - \frac{2y}{w_0}\right)|u_0|^2 - \frac{4y}{w_0}u_0^* u_1\right] - \frac{ly}{\mu\omega\rho^2}|u_0|^2, \tag{49}$$

$$S_y = -\sigma_z \frac{fk}{\mu\omega}\left[\left(\frac{lw_0 x}{\rho^2} - \frac{2x}{w_0}\right)|u_0|^2 - \frac{4x}{w_0}u_0^* u_1\right] + \frac{lx}{\mu\omega\rho^2}|u_0|^2, \tag{50}$$

$$S_z = \frac{k}{\mu\omega}|u_0|^2. \tag{51}$$

The total energy flux over the surface perpendicular to the direction of propagation is given by the integral of the Poynting vector over the surface. It is clear that there is no net energy flux in the transverse directions as $S_x$ and $S_y$ are odd $x$ and $y$. In fact, it can be seen from the general structure of the exact field equations that there cannot be any energy flux in the transverse directions. Noting that the transverse field components are even in the number of transverse variables, and the longitudinal field components are odd in the number of transverse variables, $S_x$ and $S_y$ will always be odd in transverse coordinates and thus vanish upon integration over the entire plane.

Similarly, the time averaged angular momentum density, $\mathbf{j} = \varepsilon\mu\mathbf{r}\times\mathbf{S}$, will also have transverse components that are odd in the transverse directions. This is, again, a general property of the full field equations and not dependent upon the paraxial approximation. Thus, upon calculating the flux of angular momentum density through the transverse plane, these will also be zero, and are omitted here in this paraxial calculation. The transverse component of the angular momentum density in the paraxial limit is

$$j_z = \varepsilon\frac{l}{\omega}|u_0|^2 - \varepsilon\frac{\sigma_z}{\omega}\left[\left(l - \frac{2\rho^2}{w_0^2}\right)|u_0|^2 - \frac{4\rho^2}{w_0^2}u_0^* u_1\right]. \tag{52}$$

However, this can be rewritten as

$$j_z = \varepsilon\left[\frac{l}{\omega} - \frac{\sigma_z\rho}{2\omega}\frac{\partial}{\partial\rho}\right]|u_0|^2, \tag{53}$$

which is in agreement with Allen *et al.* [24]. With this in hand, it is easy to show that the ratio of the total angular momentum flux through the plane perpendicular to the direction of the beam's travel with the energy flux through the same region, in the paraxial limit, is

$$\frac{\int j_z \mathrm{d}\rho}{\int S_z \mathrm{d}\rho} = \sqrt{\varepsilon\mu}\left(\frac{l+\sigma_z}{\omega}\right), \tag{54}$$

which is the characteristic property of paraxial Laguerre–Gaussian beams. Note that even though our definition of $u_0$ is not exactly what is used in the typical treatment of paraxial Laguerre–Gaussian beams, $|u_0|^2$ does agree with such treatments for a beam of constant waist.

While this result for paraxial beams has been known for some time, what is novel about this derivation is that it begins by solving for the full field components in terms of the small parameter $f$ and then recovering the paraxial terms, rather than simply solving the paraxial equation for only the paraxial terms. Thus, in a fully consistent manner, one can recover the complete angular momentum flux from the results presented here.

A similar process is used to derive the first nonparaxial term. First noting that $S_{z,n} \propto f^{2n}$ and $j_{z,n} \propto f^{2n}/k$, we can find the first nonparaxial term in powers of $f$, where $S_{z,n}$ is the $n$th term of the total Poynting vector component $S_z$, and $j_{z,n}$ is defined similarly. Thus, keeping terms of $O(f^2)$, we find that the total angular momentum can be expressed as

$$\frac{j_z}{S_z} = \frac{j_{z,0} + j_{z,2}}{S_{z,0} + S_{z,2}}. \quad (55)$$

Pulling out a factor of $S_{z,0}$ and performing a binomial expansion, this can be rewritten as

$$\frac{j_z}{S_z} = \frac{j_{z,0}}{S_{z,0}} + \frac{j_{z,2}}{S_{z,0}} - \frac{j_{z,0}}{S_{z,0}}\frac{S_{z,2}}{S_{z,0}}, \quad (56)$$

and the last two terms can be clearly identified as the leading order nonparaxial correction to the normalized total angular momentum. Calculating these terms is again an exercise in keeping consistent terms from the expressions for the fields and the algebra is quite lengthy. Upon integrating over the transverse plane $x^2 = y^2 = \rho^2/2$, and these terms can be written as

$$\frac{\int j_{z,2}\mathrm{d}\rho}{\int S_{z,0}\mathrm{d}\rho} = \sqrt{\varepsilon\mu}\frac{f^2}{\omega}\Bigg[(1-\sigma_z)l^2(l-1)w_0^2\int_0^\infty \frac{1}{\rho}v_{00}^2\mathrm{d}\rho - 2(1+\sigma_z)l(l-1)\int_0^\infty v_{00}v_{10}\rho\mathrm{d}\rho + 2(1+\sigma_z)\frac{l}{w_0^2}\int_0^\infty v_{10}^2\rho^3\mathrm{d}\rho + \frac{\sigma_z}{w_0^2}\int_0^\infty [v_{11}v_{00} - v_{10}v_{01}]\rho^3\mathrm{d}\rho\Bigg], \quad (57)$$

$$\frac{\int S_{z,2}\mathrm{d}\rho}{\int S_{z,0}\mathrm{d}\rho} = f^2\Bigg[(1+2l)\int_0^\infty v_{00}v_{10}\rho\mathrm{d}\rho - \int_0^\infty v_{00}v_{01}\rho\mathrm{d}\rho - \frac{2}{w_0^2}\int_0^\infty v_{00}v_{20}\rho^3\mathrm{d}\rho\Bigg]. \quad (58)$$

Note that even though one of the terms appears diverge for $\rho = 0$, it does not since $v_{00}(\rho)$ has $l$ factors of $\rho$ in it; in the case $l = 0$ the prefactor removes this term from the expression. Additionally, while the fields generated with the angular spectrum method demonstrate divergences, the $z$ dependence is lost in the calculation of the Poynting vector, which is expected physically since all of the momentum is travelling in the longitudinal direction while an averaging integration is performed over the entire transverse plane. Thus, it is plausible to conclude that this expression represents the nonparaxial angular momentum even in the far-field regime.

From these expressions, it is impossible to tell which of these factors contribute to the orbital angular momentum and which contribute to the spin angular momentum, but from evaluating these terms numerically, it can be shown that this leading nonparaxial term is not zero. It should also be noted that this term is nonzero even without the binomial expansion used for Eq. (56). This observation constitutes a direct refutation of previous results claiming that the paraxial angular momentum was the full angular momentum [4], as emphasized previously by Li [7].

## 6. SUMMARY

We summarize the main results presented above. First, the angular spectrum method was used to derive the exact solution for the full field components of Laguerre–Gaussian beams in an order expansion of the small parameter $f$. This derived expansion naturally and consistently identifies the expansion terms beyond the usual paraxial approximation, thus providing a framework to examine experimental deviations from strict paraxial response. Furthermore, the conservation of angular momentum in Laguerre–Gaussian beams arises in a transparent manner. Second, by examining the complete field solutions with arbitrary field polarization, the expected separation of angular momentum flux is recovered in the paraxial limit. Third, our analysis of the leading nonparaxial contribution to the total angular momentum demonstrates that the paraxial angular momentum is only valid within the paraxial limit. The form of the field components provides an explicit representation for a discussion of the nonparaxial contributions and extends earlier integral equation treatments, such as those in Barnett and Allen [21].

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