

# Supplementary Material

## Is single-mode lasing possible in an infinite periodic system?

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In the main text, we showed how to apply numerical stability analysis to evaluate the stability of any lasing mode for any given system. In this supplementary material, we obtain general analytical results for the specific question of stability near lasing threshold.

In particular, we use perturbation theory to compute the stability eigenvalues  $\sigma(q = q_0 + \delta k, d)$  for small  $\delta k$ , where  $D_0 = D_t(1 + d^2)$  with  $D_t$  being the pump at threshold, for points  $q_0$  where  $\sigma(q_0, 0) = 0$ . We validate our semi-analytical results against brute-force stability eigenvalues computed as in the main text, showing excellent agreement. The perturbation theory is particularly subtle due to eigenvalue crossings that result in “critical lines” where  $\sigma$  changes form, and these are also reproduced in the numerical validation. The final result is a formula that determines stability near threshold in terms of simple integrals of the threshold lasing mode. In the limit of low-loss resonances, this result further simplifies to a criterion relating band curvature to gain detuning as mentioned in the main text.

### I. PERTURBATION ANALYSIS

In all systems, we have by definition  $\sigma(0, 0) = 0$ . For reciprocal systems, the mode at  $-k$  also reaches threshold at  $D_t$  so that  $\sigma(\pm 2k, 0) = 0$  [1]. Note that this last case does not have to be considered when  $k$  and  $-k$  are separated with lattice vectors, as for example when lasing at a band edge or at the center of the Brillouin zone. We first give a detailed derivation in the case  $q_0 = 0$ , and then present the results for  $q_0 = \pm 2k$ .

The stability eigenproblem is given by  $(A_q + B\sigma + C\sigma^2)U_q = 0$ , where:

$$A_q = \begin{pmatrix} \Delta_{k,q}^r & -\Delta_{k,q}^i & \omega^2 & 0 & 0 \\ \Delta_{k,q}^i & \Delta_{k,q}^r & 0 & \omega^2 & 0 \\ \gamma_{\perp} D & 0 & \omega_a - \omega & \gamma_{\perp} & \gamma_{\perp} \mathbf{E}^r \\ 0 & \gamma_{\perp} D & -\gamma_{\perp} & \omega_a - \omega & \gamma_{\perp} \mathbf{E}^i \\ -\gamma_{\parallel} \mathbf{P}^i & \gamma_{\parallel} \mathbf{P}^r & \gamma_{\parallel} \mathbf{E}^i & -\gamma_{\parallel} \mathbf{E}^r & \gamma_{\parallel} \end{pmatrix}, B = \begin{pmatrix} -\sigma_c & -2\epsilon_c \omega & 0 & -2\omega & 0 \\ 2\epsilon_c \omega & -\sigma_c & 2\omega & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} -\epsilon_c & 0 & -1 & 0 & 0 \\ 0 & -\epsilon_c & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{S1})$$

with  $\Delta_{k,q}^r = -e^{-iqx} \text{Re}(\Theta_k) e^{iqx} + \epsilon_c \omega^2$ ,  $\Delta_{k,q}^i = -e^{-iqx} \text{Im}(\Theta_k) e^{iqx} + \sigma_c \omega$ ,  $\mathbf{E}^r = \text{Re}(\mathbf{E})$  and  $\mathbf{E}^i = \text{Im}(\mathbf{E})$ . For brevity of notation, we removed the subscript  $k$  from  $\omega_k$ ,  $\mathbf{E}_k$ ,  $\mathbf{P}_k$ ,  $D_k$ , but vectors still refer to the periodic part of Bloch terms. The SALT mode can be expanded in  $d$ , as for example done in Ref. 1. In particular, we have:

$$\omega \approx \omega_t + \omega_2 d^2, \quad \mathbf{E} \approx d \frac{a \mathbf{E}_+}{\Gamma_t}, \quad |a|^2 = \frac{G_D + \omega_2 H}{I}, \quad \omega_2 = -\text{Im} \left( \frac{G_D}{I} \right) / \text{Im} \left( \frac{H}{I} \right) \quad (\text{S2})$$

where  $\mathbf{E}_+$  (resp.  $\mathbf{E}_-$ ) is a solution to the linear SALT equation at threshold with Bloch vector  $k$  (resp.  $-k$ ).  $G_D$ ,  $I$  and  $H$  are given by:

$$G_C = \int d\mathbf{x} (\epsilon_c + i\sigma_c \omega_t) \mathbf{E}_- \cdot \mathbf{E}_+, \quad G_D = \int d\mathbf{x} D_t \mathbf{E}_- \cdot \mathbf{E}_+, \quad I = \int d\mathbf{x} D_t |\mathbf{E}_+|^2 \mathbf{E}_- \cdot \mathbf{E}_+, \quad H = \frac{1}{\omega_t^2 \Gamma_t} \frac{\partial}{\partial \omega_t} [\omega_t^2 (G_C + G_D \Gamma_t)]. \quad (\text{S3})$$

Note that there is an arbitrary choice for the phase of  $a$ . To simplify some computations, we take  $a \Gamma_t^*$  to be real.

Operators  $A_q$ ,  $B$  and  $C$  can then be expanded in  $(\delta k = q - q_0, d)$ :

$$A_q \approx A_{00} + A_{01} d + A_{02} d^2 + A_{10} \delta k + A_{20} \delta k^2, \quad B \approx B_0 + B_2 d^2, \quad C = C_0. \quad (\text{S4})$$

As a result, eigenvalues and eigenvectors can be expanded in the same way:

$$U_q \approx \sum_{i,j \leq 2} U_{ij} \delta k^i d^j, \quad \sigma \approx \sum_{i,j \leq 2} \sigma_{ij} \delta k^i d^j. \quad (\text{S5})$$

A crucial point that we confirm later, is that  $\sigma$  is *not* necessarily analytical at  $(q_0, 0)$  since there is a degeneracy. So equation (S5) is not valid inside a ball around  $(\delta k, d) = (0, 0)$ . Instead, we have different expansion coefficients depending on the path  $(\delta k, d)$ .

We first consider  $q_0 = 0$ . The zeroth-order stability problem is equivalent to the threshold SALT equation at  $k$ . Because real and imaginary parts of the field are split, we have two degenerate eigenvectors  $v_p$  corresponding to  $\sigma_{00} = 0$ , where:

$$v_p = (\text{Re}(\mathbf{e}_p^+), \text{Im}(\mathbf{e}_p^+), D_t \text{Re}(\Gamma_t \mathbf{e}_p^+), D_t \text{Im}(\Gamma_t \mathbf{e}_p^+), 0), \quad (\text{S6})$$

for  $\mathbf{e}_{1,2}^+ = \mathbf{E}_+, i\mathbf{E}_+$ . We also need solutions  $w_p$  to the transverse problem  $w_p^t A_{00} = 0$  given by:

$$w_p = \left( \text{Re}(\mathbf{e}_p^-), -\text{Im}(\mathbf{e}_p^-), \frac{\omega_t^2}{\gamma_\perp} \text{Re}(\Gamma_t \mathbf{e}_p^-), -\frac{\omega_t^2}{\gamma_\perp} \text{Im}(\Gamma_t \mathbf{e}_p^-), 0 \right), \quad (\text{S7})$$

where  $\mathbf{e}_{1,2}^- = \mathbf{E}_-, i\mathbf{E}_-$ .

We now have  $U_{00} = b_1 v_1 + b_2 v_2$ , where  $b_p$  are to be determined by degenerate perturbation theory. As we will see later, the coefficients  $b_p$  depend on the path  $(\delta k, d)$ . To simplify notations, we note  $M = [w_j^t M v_p]_{jp}$  for a given operator matrix  $M$ . The first order perturbation equations are given by:

$$\begin{aligned} (\delta k) \quad & (B_0 \sigma_{10} + A_{10}) U_{00} + A_{00} U_{10} = 0 \rightarrow \bar{A}_{10} b = -\sigma_{10} \bar{B}_0 b \\ (d) \quad & (B_0 \sigma_{01} + A_{01}) U_{00} + A_{00} U_{01} = 0 \rightarrow \bar{A}_{01} b = -\sigma_{01} \bar{B}_0 b. \end{aligned} \quad (\text{S8})$$

It is straightforward to show that  $\bar{A}_{01} = 0$ ,  $\bar{B}_0 = -\text{Im}(\omega_t^2 \Gamma_t H M)$  and  $\bar{A}_{10} = i \text{Im}(LM)$ , where  $M = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$  and  $L = -\int d\mathbf{x} \mathbf{E}_- \cdot \partial_q \Theta_{k+q} \mathbf{E}_+$  (in particular,  $-\partial_q \Theta_{k+q} = 2i e^{-ikx} \nabla e^{ikx}$  for  $\mathbf{E} = \mathbf{E}_z \mathbf{z}$  waves). We then have:

$$\sigma_{01} = 0, \quad \sigma_{10} = i \frac{L}{\omega_t^2 \Gamma_t H} \text{ or } \sigma_{10} = i \left( \frac{L}{\omega_t^2 \Gamma_t H} \right)^*. \quad (\text{S9})$$

Since 0 is a maximum of  $\text{Re}[\sigma(\delta k, 0)]$ ,  $\sigma_{01}$  is purely imaginary and the two eigenvalues are identical. So  $\bar{A}_{10} + \sigma_{10} \bar{B}_0 = 0$  and  $b$  is not determined by first order equations. Note that  $i\sigma_{10}$  is simply the slope of  $\omega(k)$  at the lasing  $k$ . We can also see that:

$$U_{01} = -\sum b_p g_p + \sum c_l v_l, \quad U_{10} = -\sum b_p A_{00}^{-1} (\sigma_{10} B_0 + A_{10}) v_p + \sum \tilde{c}_l v_l, \quad (\text{S10})$$

where  $g_p^5 = 2D_t \text{Re}(\Gamma_t a^* \mathbf{e}_p^+ \cdot \mathbf{E}_+^*)$  and the first fourth components of  $g_p$  are zero.  $c_l$  and  $\tilde{c}_l$  are arbitrary complex coefficients that will not affect our results. Note also that the fifth component of  $U_{10}$  is equal to zero.

The second order perturbation equations are now given by:

$$\begin{aligned} (\delta kd) \quad & \sigma_{11} B_0 U_{00} + (A_{10} + \sigma_{10} B_0) U_{01} + A_{01} U_{10} + A_{00} U_{11} = 0 \\ (\delta k^2) \quad & (A_{20} + \sigma_{20} B_0 + \sigma_{10}^2 C) U_{00} + (A_{10} + \sigma_{10} B_0) U_{10} + A_{00} U_{20} = 0 \\ (d^2) \quad & (A_{02} + \sigma_{02} B_0) U_{00} + A_{01} U_{01} + A_{00} U_{02} = 0. \end{aligned} \quad (\text{S11})$$

We start by solving the three equations independently. From results of first-order perturbation we can see that  $w_j^t (A_{10} + \sigma_{10} B_0) U_{01} = 0$  and  $w_j^t A_{01} U_{10} = 0$ . The equation of order  $\delta kd$  then gives  $\sigma_{11} = 0$ .

Multiplying the equation of order  $\delta k^2$  by  $w_j^t$  we get:

$$-\sigma_{20} \bar{B}_0 b = (\bar{A}_{20} + \sigma_{10}^2 \bar{C} + \bar{P}) b = \text{Re}(XM) b, \quad \text{where } P = (\sigma_{10} B_0 + A_{10}) A_{00}^{-1} (\sigma_{10} B_0 + A_{10}), \quad (\text{S12})$$

where eigenvalues are simply related to the curvature of  $\omega(k)$  at the lasing  $k$  ( $= i\sigma_{20}$ ):

$$\sigma_{20} = i \frac{X}{\omega_t^2 \Gamma_t H} \text{ or } \sigma_{20} = -i \left( \frac{X}{\omega_t^2 \Gamma_t H} \right)^*, \quad b = (1, \mp i). \quad (\text{S13})$$

The degeneracy is artificially due to the separation of the real and imaginary parts of the field, so  $X$  can be easily recovered from the non-degenerate perturbation theory of  $\omega(k)$  in  $k$ . We obtain:

$$X = \int d\mathbf{x} \mathbf{E}_- \cdot \square \mathbf{E}_+, \quad \square = \partial_q^2 \Theta_{k+q} - \frac{\sigma_{10}^2}{2} \partial^2 G + (i\partial_q \Theta_{k+q} + \sigma_{10} \partial G)(-\Theta_k + G)^{-1} (i\partial_q \Theta_{k+q} + \sigma_{10} \partial G), \quad (\text{S14})$$

$$G(\omega_t) = \omega_t^2 \left[ \epsilon_c + i \frac{\sigma_c}{\omega_t} + D_t \Gamma(\omega_t) \right] \quad \text{and} \quad \partial_q^2 \Theta_{k+q} = -I \quad \text{for} \quad \mathbf{E} = \bar{\mathbf{E}}_2 \mathbf{z} \quad \text{waves.}$$

Finally, multiplying the equation of order  $d^2$  by  $w_j^t$  we get (using  $a\Gamma_t^* = a^* \Gamma_t$ ):

$$-\sigma_{02} \bar{B}_0 b = (\bar{A}_{02} - \bar{Q}) b, \quad \text{with} \quad \bar{Q} = [w_j^t A_{01} g_p]_{jp} = \text{Re} [\omega_t^2 \Gamma_t |a|^2 I (M' + M)] \quad \text{and} \quad \bar{A}_{02} = 0, \quad (\text{S15})$$

where  $M' = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ . The eigenvalues are then given by:

$$\sigma_{02} = 0, \quad b = (0, 1) \quad \text{or} \quad \sigma_{02} = 2|a|^2 \text{Im} \left( \frac{I}{H} \right), \quad b = (-\text{Im}[I/H], \text{Re}[I/H]). \quad (\text{S16})$$

We see that we obtain different eigenvectors in (S13) and (S16). This means that the expansion in (S5) depends on the path  $(\delta k, d)$ . If  $d = o(\delta k)$ , the expansion is determined by (S13); while it is determined by (S16) if  $\delta k = o(d)$ . A critical behaviour is obtained along the line  $\delta k = \alpha d$  for which the second order term is given by  $\sigma_2 d^2$  and the three equations in (S11) have to be combined. In this case, the second order perturbation eigenproblem becomes:

$$-\sigma_2 \bar{B}_0 b = [\alpha^2 \text{Re}(XM) - \bar{Q}] b, \quad (\text{S17})$$

and the eigenvalues are given by:

$$\sigma_2 = \text{Im}(\alpha^2 \theta + \eta_I) \pm \sqrt{|\eta_I|^2 - [\text{Re}(\alpha^2 \theta + \eta_I)]^2}, \quad \theta = -\frac{X}{\omega_t^2 \Gamma_t H}, \quad \eta_I = |a|^2 \frac{I}{H}. \quad (\text{S18})$$

Note that  $\theta$  is simply the band curvature at threshold ( $\omega(k) \approx \omega_t + i\sigma_{10} \delta k + \theta \delta k^2$ ).

The presence of the square root function clearly shows the non-analyticity of  $\sigma$ . In particular, there is an eigenvalue crossing for  $\alpha_c^2 = (-\text{Re}(\eta_I) \pm |\eta_I|) / \text{Re}(\theta)$ . The stability condition ( $\sigma_2 \leq 0$ ) can also be immediately retrieved:

$$\boxed{\alpha_s^2 = -2\text{Re}(\eta_I/\theta) \leq 0.} \quad (\text{S19})$$

We can simplify the stability condition in the limit of small loss. In this case,  $H \approx 2\omega_t \int \epsilon_c \mathbf{E}_- \cdot \mathbf{E}_+ / \Gamma_t$ ,  $\mathbf{E}_- \approx \mathbf{E}_+^*$  and  $\text{Im}(\theta) \approx 0$ . The stability condition  $\text{Re}(\eta_I) \text{Re}(\theta) + \text{Im}(\eta_I) \text{Im}(\theta) \geq 0$  becomes equivalent to:

$$\boxed{\text{Re}(\theta) (\omega_t - \omega_a) \gtrsim 0.} \quad (\text{S20})$$

This means that the sign of the detuning  $(\omega_t - \omega_a)$  should be the same as the sign of the band curvature  $(\text{Re}[\theta])$ . For example, when lasing at a bandedge, this means that  $\omega_a$  should be inside the bandgap.

As mentioned in the beginning of the section, in the case of degenerate lasing, the previous analysis should also be carried out at  $q_0 = -2k$  (or equivalently at  $2k$ ). (Note that we are not considering the special case of a degeneracy that comes for a wavevector other than  $-k$ . However, this situation can be studied in a similar way by computing a perturbation expansion of  $\sigma$  around multiple adequate  $q_{0s}$ .) It is easy to see that the solutions of the zeroth order problem  $A_{-2k} U_{00} = 0$  are related to solutions of SALT at  $k \pm 2k$ . Two separate cases should then be considered.

*a.*  $ka = \pi/2$ : In this case, the problems at  $-k$  and  $3k$  are equivalent (separated by a lattice vector) and the zeroth order problem is degenerate. The eigenvectors are given by:

$$v_p = e^{i\pi x/a} \left( \text{Re} \left( e^{-i\pi x/a} \mathbf{e}_p^- \right), \text{Im} \left( e^{-i\pi x/a} \mathbf{e}_p^- \right), D_t \text{Re} \left( e^{-i\pi x/a} \Gamma_t \mathbf{e}_p^- \right), D_t \text{Im} \left( e^{-i\pi x/a} \Gamma_t \mathbf{e}_p^- \right), 0 \right), \quad (\text{S21})$$

while solutions of the transverse problem become:

$$w_p = e^{-i\pi x/a} \left( \text{Re} \left( e^{i\pi x/a} \mathbf{e}_p^+ \right), -\text{Im} \left( e^{i\pi x/a} \mathbf{e}_p^+ \right), \frac{\omega_t^2}{\gamma_{\perp}} \text{Re} \left( e^{i\pi x/a} \Gamma_t \mathbf{e}_p^+ \right), -\frac{\omega_t^2}{\gamma_{\perp}} \text{Im} \left( e^{i\pi x/a} \Gamma_t \mathbf{e}_p^+ \right), 0 \right). \quad (\text{S22})$$

We now have  $g_p^5 = 2D_t e^{i\pi x/a} \text{Re}(\Gamma_t a^* e^{-i\pi x/a} \mathbf{e}_p^- \cdot \mathbf{E}_+^*)$  and  $\bar{Q} = \text{Re}[\omega_t^2 \Gamma_t |a|^2 (KM' + JM)]$ , where:

$$J = \int d\mathbf{x} D_t (\mathbf{E}_+^* \cdot \mathbf{E}_-) (\mathbf{E}_+ \cdot \mathbf{E}_+) \text{ and } K = \int d\mathbf{x} e^{2i\pi x/a} D_t (\mathbf{E}_-^* \cdot \mathbf{E}_+) (\mathbf{E}_+ \cdot \mathbf{E}_+). \quad (\text{S23})$$

We can then obtain the eigenvalues of the problem (S17) for  $\delta k = q + 2k = \alpha d$ :

$$\sigma_2 = \text{Im}(\alpha^2 \theta + \eta_J) \pm \sqrt{|\eta_K|^2 - [\text{Re}(\alpha^2 \theta + \eta_J)]^2}, \quad \eta_J = |a|^2 \frac{J}{H}, \quad \eta_K = |a|^2 \frac{K}{H}. \quad (\text{S24})$$

The stability condition is now equivalent to:

$$\boxed{\alpha_s^2 = -\text{Re}\left(\frac{\eta_J}{\theta}\right) + \sqrt{\left|\frac{\eta_K}{\theta}\right|^2 - \left|\frac{\eta_J}{\theta}\right|^2 + \text{Re}\left(\frac{\eta_J}{\theta}\right)^2} \text{ non-real or real negative.}} \quad (\text{S25})$$

*b.  $ka \neq \pi/2$ :* In this case, the problems at  $-k$  and  $3k$  are different, and only  $-k$  has a solution. The zeroth order problem for  $q_0 = -2k$  is now *not* degenerate and eigenvectors are given by:

$$v = (1, -i, D_t \Gamma_t, -i D_t \Gamma_t, 0) \mathbf{E}_-, \quad w = \left(1, i, \frac{\omega_t^2 \Gamma_t}{\gamma_\perp}, i \frac{\omega_t^2 \Gamma_t}{\gamma_\perp}, 0\right) \mathbf{E}_+. \quad (\text{S26})$$

The dimension of our problem is now one and we have  $g^5 = 2D_t \Gamma_t a^* \mathbf{E}_+^* \cdot \mathbf{E}_-$ ,  $\bar{B}_0 = 2i\omega_t^2 \Gamma_t H$ ,  $A_{20} = 2X$  and  $\bar{Q} = 2\omega_t^2 \Gamma_t |a|^2 J$ . The unique eigenvalue of (S17) is now equal to:

$$\sigma_2 = -i(\theta \alpha^2 + \eta_J). \quad (\text{S27})$$

This simply means that there is no eigenvalue crossing and that the expansion of  $\sigma$  does not depend on the path  $(\delta k, d)$ . Note that  $\sigma_2^*$  is also an eigenvalue around  $q_0 = 2k$  (which is simply due to the facts that our operators  $A, B$  and  $C$  are real as indicated in the main text). The stability condition is immediately given by:

$$\boxed{\text{Im}(\eta_J) \leq 0}, \quad (\text{S28})$$

since we already have  $\text{Im}(\theta) \leq 0$  ( $\text{Im}[\omega(k)]$  has a maximum at  $k$ ). Note that this stability condition is equivalent to having a stable lasing close to threshold for the *single* unit-cell problem.

Finally, some useful points to mention:

- We have  $\eta_I = G_D/H + \omega_2$ . It is also straightforward to use perturbation theory to show that  $\omega_2^l = -G_D/H$  where  $\omega_2^l$  is the slope (in  $D_0/D_t - 1$ ) of the eigenfrequency of the linear problem at the threshold *without* gain saturation ( $\omega^l \approx \omega_t + \omega_2^l (D_0/D_t - 1)$ ). By definition, threshold should be reached from below the real axis, so  $\text{Im}(\omega_2^l) \geq 0$ . Since  $\omega_2$  is real, we conclude that  $\text{Im}(\eta_I) = -\text{Im}(\omega_2^l) \leq 0$ . This means that  $\sigma_{02} \leq 0$  and that the single unit-cell lasing problem is always stable near threshold in absence of degeneracy.
- For TM waves ( $\mathbf{E} = E_z \mathbf{z}$ ), we have  $I = J$ . This means that  $\text{Im}(\eta_J) \leq 0$  and that the single unit-cell lasing problem is also stable in the degenerate case when  $k \neq \pi/2$ . This is an analytical proof for part of the stability result conjectured in Ref. 1. Note that  $k = \pi/2$  is equivalent to the condition  $n = 4\ell$  in Ref. 1.
- For TM waves and  $k \neq \pi/2$ , we conclude that  $\sigma_2 \leq 0$  when expanding around  $-2k$ . So the stability is only determined by the expansion around 0 ( $-\text{Re}(\eta_I/\theta) \leq 0$ ).

## II. NUMERICAL VALIDATION

Here, we present a numerical validation of the analytical perturbation-theory results discussed in the previous section.

Figure S1 shows results for the 1d structure studied in the main text. Figs. S2–S3 are for the same structure, but with  $\omega_a$  lying below the lasing band edge, outside the bandgap, leading to instability near threshold as predicted above. In both cases, the numerical simulations show near-perfect agreement with the analytical results.

Figures S5–S4 show results for the 2d structures presented in the main text with a positive and negative laser detuning, respectively. Again, numerical simulations are in agreement with the analytical results.

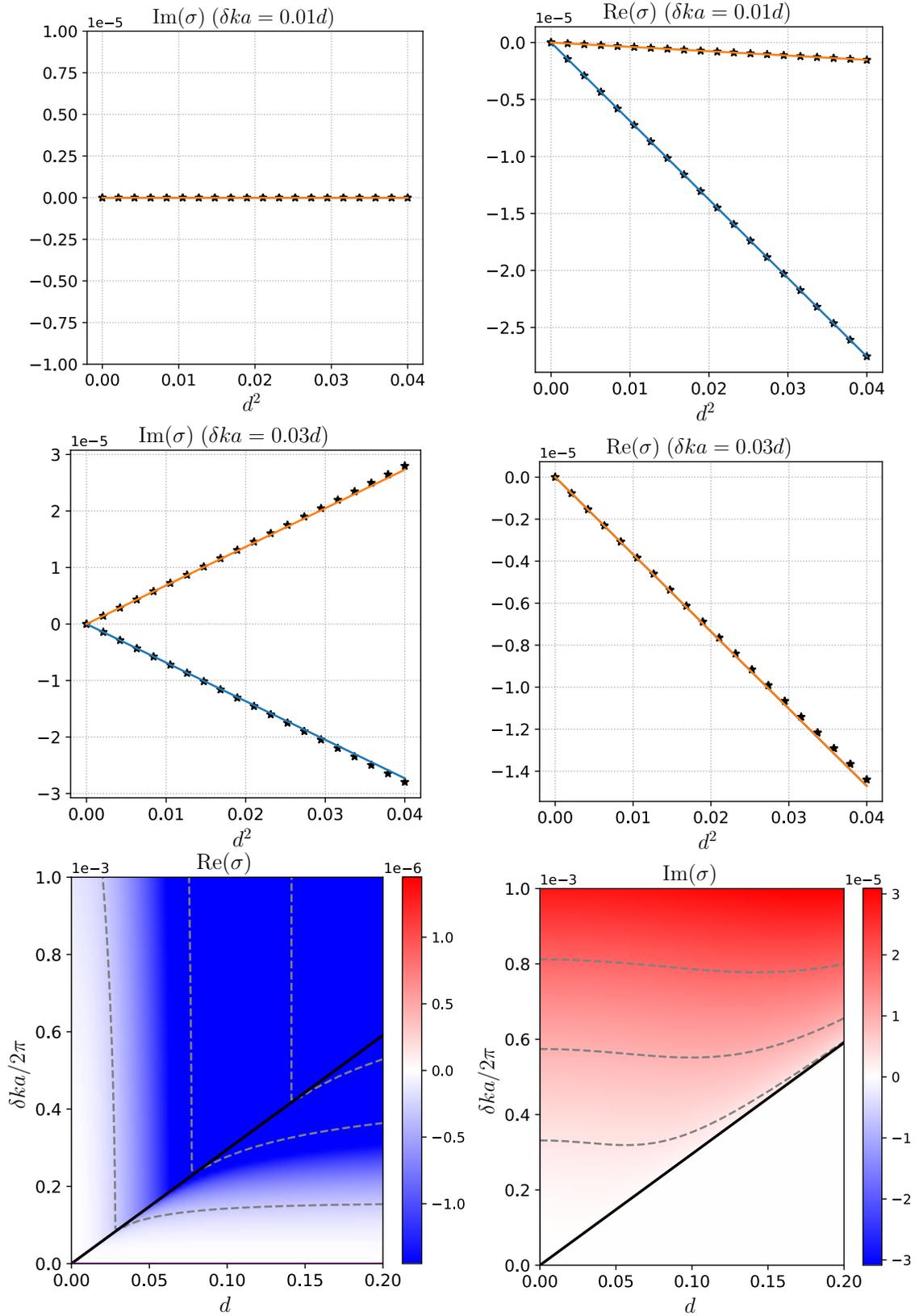


FIG. S1. Same 1d structure in the main text. Numerical simulation (stars and dashed contour lines) are in agreement with analytical results (solid lines). Since the lasing mode is at a bandedge, we have  $\sigma_{10} = 0$ . Black line corresponds to  $\delta ka = \alpha_c d$  and represents the line of eigenvalue crossing (transition from two real to two complex conjugate eigenvalues).  $\alpha_c \approx 0.018$  and  $\alpha_s^2 \approx -4.2 \times 10^{-4}$ .

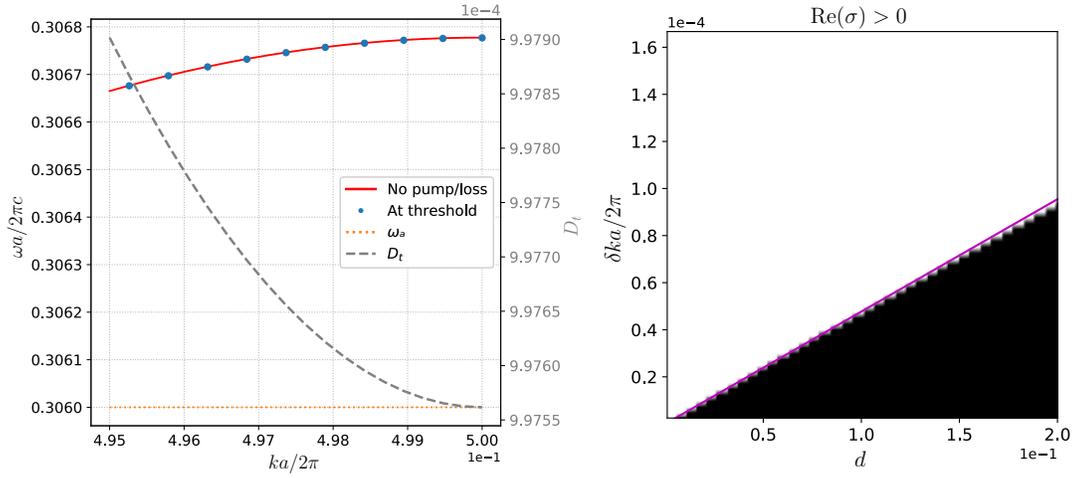


FIG. S2. Same 1d structure in the main text but with  $\omega_a a / 2\pi c = 0.306$  and  $\gamma_{\perp} / 2\pi c = 0.08$ . The lasing mode is still at the band edge but the laser detuning ( $\omega_t - \omega_a$ ) is now positive.

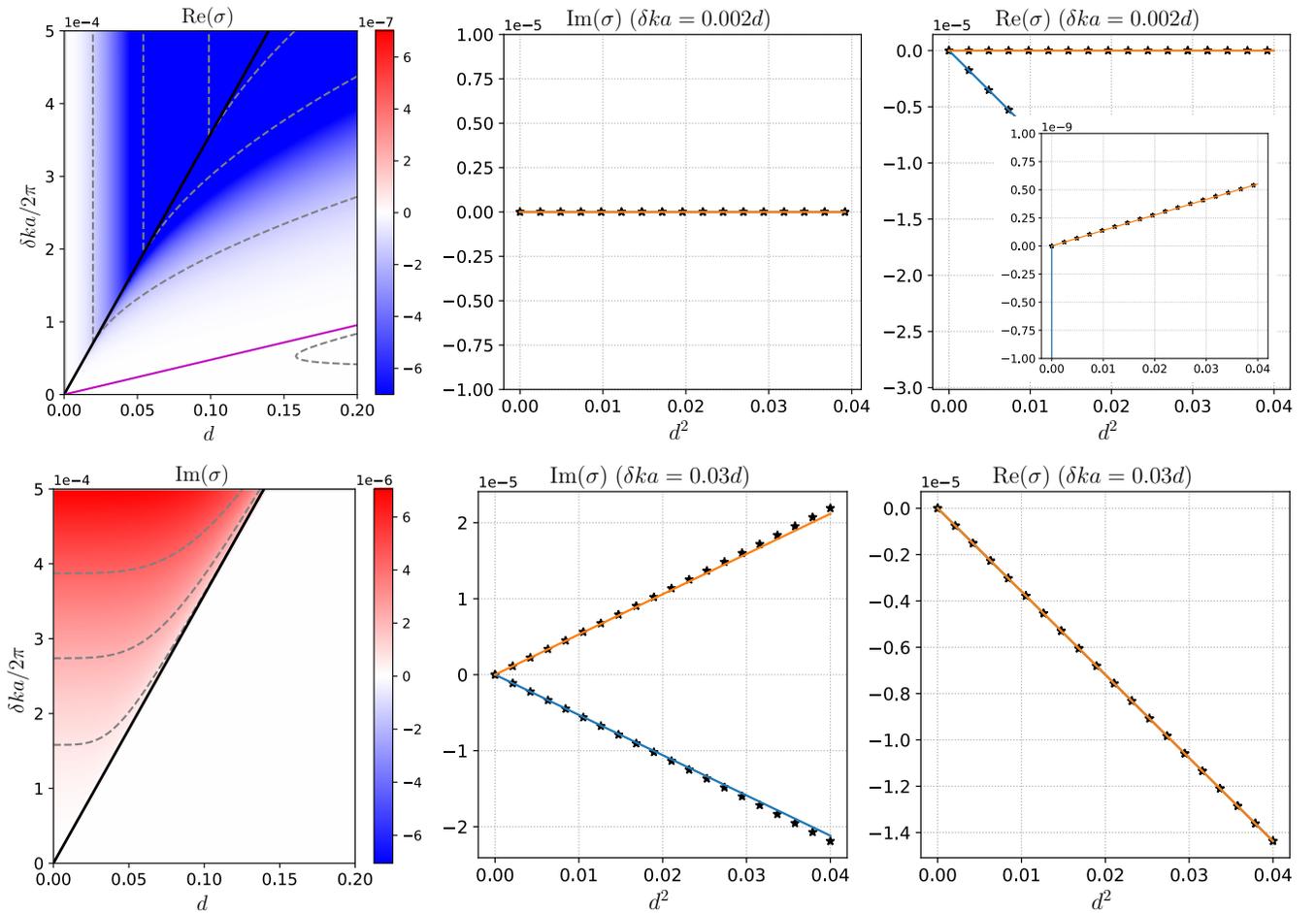


FIG. S3. Same 1d structure studied in the main text but with  $\omega_a a / 2\pi c = 0.306$  and  $\gamma_{\perp} / 2\pi c = 0.08$ . Numerical simulation (stars and dashed contour lines) are in agreement with analytical results (solid lines). Black line corresponds to  $\delta ka = \alpha_c d$  and represents the line of eigenvalue crossing (transition from two real to two complex conjugate eigenvalues). Magenta solid line corresponds to  $\delta k = \alpha_s d$  from analytical perturbation results and matches  $\text{Re}(\sigma) = 0$  from numerical simulation.  $\alpha_c \approx 0.022$  and  $\alpha_s \approx 3 \times 10^{-3}$ .

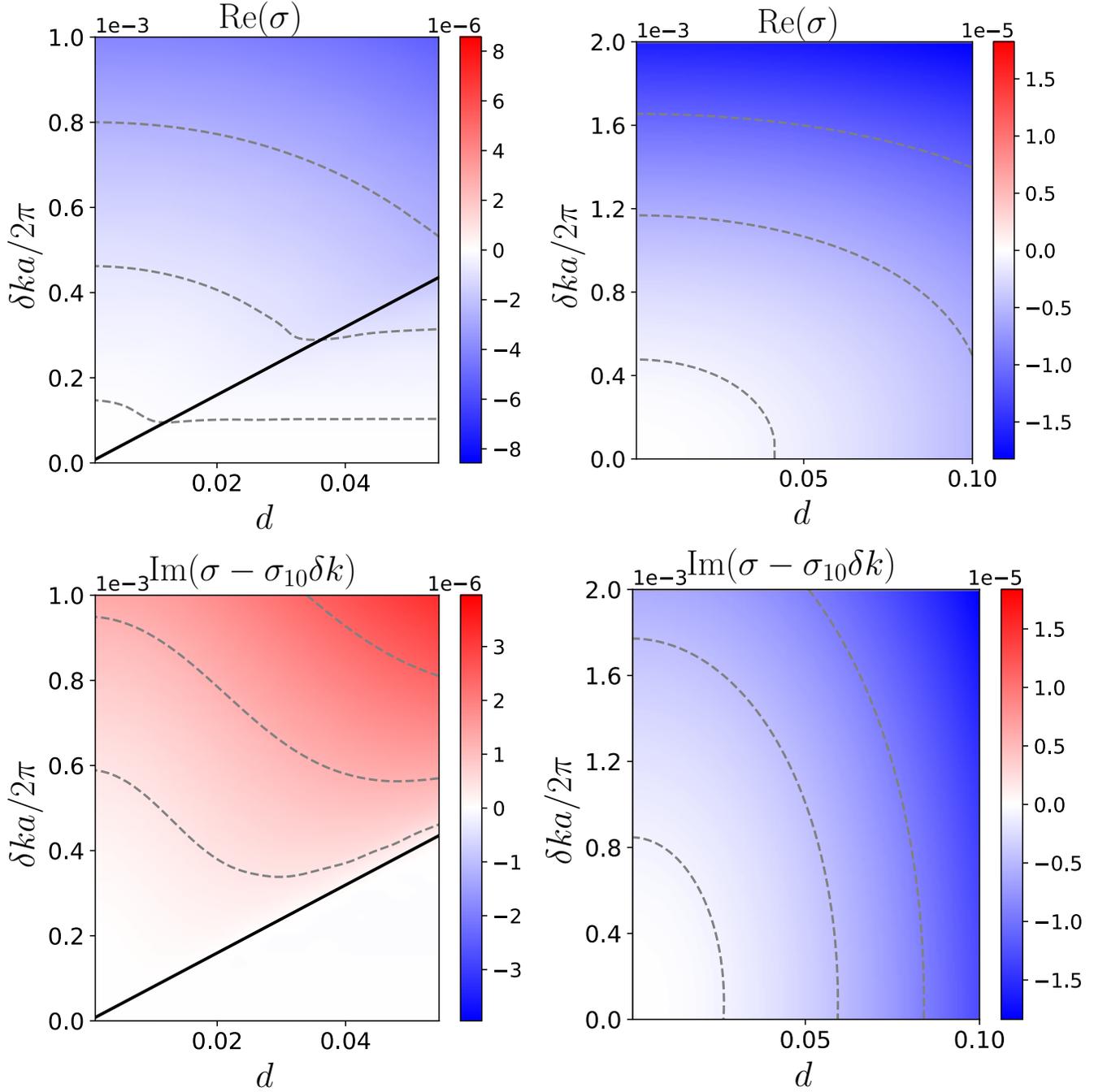


FIG. S4. Same 2d structure in the main text with  $\omega_a a/2\pi c = 0.625$  and  $\omega_t a/2\pi c \approx 0.65$ . Left:  $q_0 = 0$ . Right:  $q_0 = -2k$ . Contour lines (dashed) are from numerical simulation. Black solid line corresponds to  $\delta k = \alpha_c d$  from analytical perturbation results and represents the line of eigenvalue crossing (transition of  $\sigma - \sigma_{10}\delta k$  from two real to two complex conjugate eigenvalues) when expanding around  $q_0 = 0$ . The analytical line matches results of numerical simulation. Expansion around  $-2k$  does not show a critical line in agreement with perturbation theory (case  $ka \neq \pi/2$ ). We have  $\alpha_c \approx 0.05$ ,  $\alpha_s^2 \approx -0.018$  and  $\sigma_{10} \approx 0.59i$  when expanding around  $q_0 = 0$  (opposite sign for  $i\sigma_{10}$  when expanding around  $q_0 = -2k$ ).

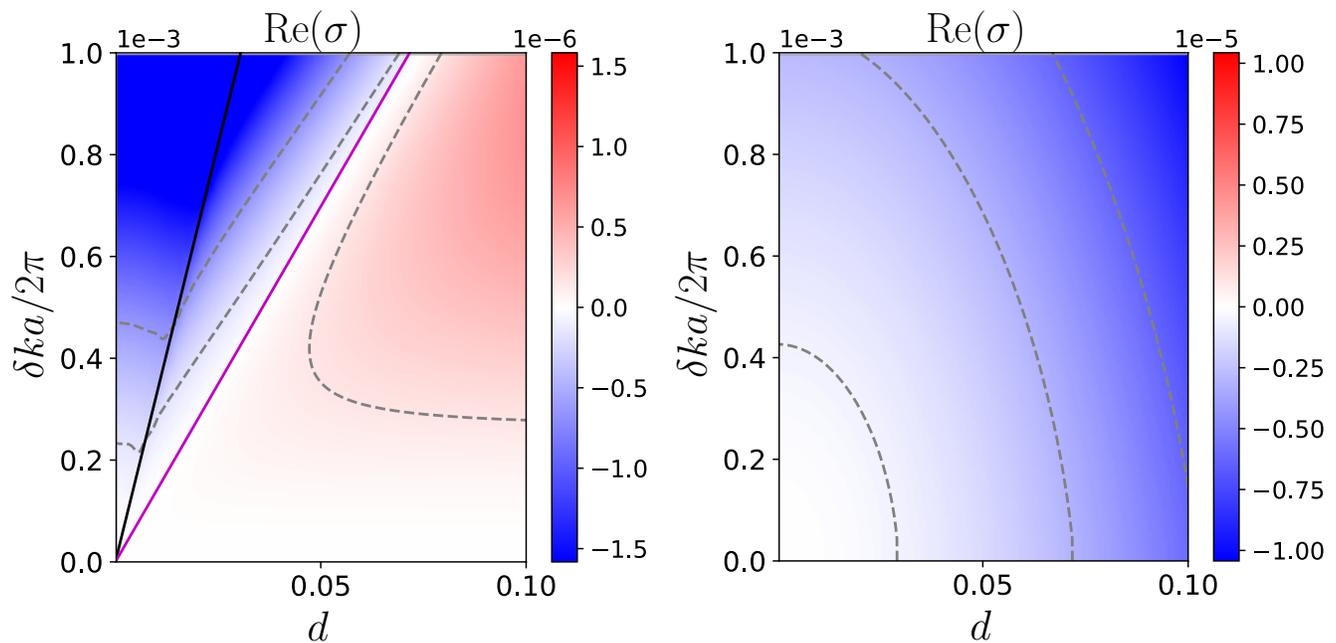


FIG. S5. Same 2d structure in the main text with  $\omega_a a/2\pi c = 0.675$ . Left:  $q_0 = 0$ . Right:  $q_0 = -2k$ . The lasing mode is slightly shifted to  $ka/2\pi \approx 0.1944$  but still with  $\omega_t a/2\pi c \approx 0.65$ . Contour lines (dashed) are from numerical simulation. Black solid line corresponds to  $\delta k = \alpha_c d$  and magenta solid line corresponds to  $\delta k = \alpha_s d$  from analytical perturbation results when expanding around  $q_0 = 0$ . Magenta line (analytical) matches  $\text{Re}(\sigma) = 0$  from numerical simulation. Expansion around  $-2k$  does not show a critical line in agreement with perturbation theory (case  $ka \neq \pi/2$ ). We have  $\alpha_c \approx 0.21$ ,  $\alpha_s \approx 0.088$  and  $\sigma_{10} \approx 0.59i$  when expanding around  $q_0 = 0$  (opposite sign for  $i\sigma_{10}$  when expanding around  $q_0 = -2k$ ).

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